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On Badly Approximable Functions and Uniform Algebras*

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1. INTRODUCTION

Let X be a compact Hausdorff space and A a complex algebra of continuous functions on X which is uniformly closed, separates points, and contains the constants (that is, a *uniform algebra*). Let $C(X)$ denote the space of continuous complex-valued functions on X. The distance from a function $\phi \in C(X)$ to A is defined to be

$$d(\phi, A) = \inf\{\| \phi - f \| : f \in A\},$$

where $\| \cdot \|$ $(= \| \cdot \|_X)$ is the supremum of absolute value over X. In this paper, we consider the problems of existence and description of the functions $\phi \in C(X)$ which satisfy

$$\| \phi \| = d(\phi, A).$$

Such a function, if not identically zero, is said to be *badly approximable with respect to A* and we write $\phi \in ba(A)$.

Our aim in this paper is threefold. First, we investigate, for certain classical algebras on sets in the complex plane, whether the set $ba(A)$ determines the algebra A. In Section 3, this is proved to be the case for $P(K)$, $R(K)$ $A(K)$, and $C(K)$ in the sense that any two of these coincide if the corresponding sets of badly approximable functions coincide. (These algebras are defined in Section 3.)

Second, we characterize algebras A for which $ba(A)$ is empty. This is Theorem 4.7. The methods used to prove it yield, as a by-product, a characterization of Dirichlet algebras in terms of badly approximable functions.

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Third, we extend the following theorems of Poreda [10] and Gamelin *et al.* [4] to a more general setting.

POREDA'S THEOREM. *Suppose $X \subset \mathbb{C}$ consists of a simple closed Jordan curve. Then $\phi \in C(X)$ is badly approximable with respect to $P(X)$ if and only if ϕ has nonzero constant modulus and* $\operatorname{ind}(\phi) < 0$.

Here and in the following theorem $\operatorname{ind}(\phi)$ may be defined as $1/2\pi$ times the change in argument of $\phi(z)$ as z travels once along X in the positive direction.

GGR AND S'S THEOREM. *Suppose that $Y \subset \mathbb{C}$ is compact and connected, and $\mathbb{C} - Y$ has finitely many components. Suppose X, the boundary of Y, consists of $N + 1$ disjoint closed Jordan curves. If $\phi \in C(X)$ is badly approximable with respect to $R(Y)|_X$ then ϕ has nonzero constant modulus and* $\operatorname{ind}(\phi) < N$.

2. PRELIMINARY NOTIONS

We will adopt the notation and definitions of Gamelin's book [3]. Let A be a uniform algebra on the compact Hausdorff space X. Let M_A denote the *maximal ideal space* of A and ∂_A its *Shilov boundary*. It is always assumed that $X \subset M_A$ by identifying a point $x \in X$ with the homomorphism of evaluation at x. Denote by A^{-1} the invertible elements of A. If $f \in A$, let $\hat{f} \in C(M_A)$ be defined by

$$\hat{f}(\beta) = \beta(f), \qquad \beta \in M_A.$$

Let $\operatorname{Re}(A)$ denote the set of real parts of functions in A. Let $\operatorname{Re}(C(X))$ be written $C_R(X)$. Then A is said to be *Dirichlet on X* if $\operatorname{Re}(A)$ is uniformly dense in $C_R(X)$.

We will use the word "measure" to mean a complex regular Borel measure. Denote by $A^\perp$ the set of measures μ on X that satisfy $\int f\,d\mu = 0$ for every $f \in A$. Let $\beta \in M_A$. A probability measure m is called a *representing measure* for β if $\int f\,dm = \beta(f)$ for all $f \in A$. Let M_β denote the set of representing measures for β. For any measure μ on X, let $\operatorname{supp}(\mu)$ be its support, let $|\mu|$ be the total variation measure, and let $\|\mu\| = |\mu|(X)$.

We recall the Arens–Royden theorem [3, p. 89]: *If $\phi \in C(M_A)$ does not vanish on M_A, then there exists $g \in A^{-1}$ such that $\phi/\hat{g}$ has a continuous logarithm on M_A.*

If $\phi \in C(X)$ let $E(\phi)$ be the set of $x \in X$ such that $|\phi(x)| = \|\phi\|$.

For a convex set Q in some vector space, let Q^e be the set of extreme points of Q. If B is a Banach space, let $\operatorname{ball}(B)$ denote its closed unit ball.

The following two theorems are simply rephrasings of well-known results in approximation theory (see [12, pp. 29 and 69]).

2.1. THEOREM. *If $\phi \in C(X)$, then $\phi \in ba(A)$ if and only if there exists a nonzero measure $\mu \in A^{\perp}$ such that* $\operatorname{supp}(\mu) \in E(\phi)$ *and* $\phi\mu \geqslant 0$. *The measure μ may be chosen from* $\operatorname{ball}(A^{\perp})^e$.

2.2. THEOREM. *If $\phi \in C(X)$, then $\phi \notin ba(A)$ if and only if there exists $f \in A$ such that*

$$\operatorname{Re} f(x)\, \overline{\phi(x)} > 0, \qquad \text{for all } x \in E(\phi).$$

If F is a closed subset of X, let $A\,|_F$ denote the set of restrictions to F of the functions in A, and let A_F be its closure in $C(F)$. By either of these two theorems, $\phi \in ba(A)$ if and only if $\phi\,|_F \in ba(A_F)$ for some closed set $F \subset E(\phi)$. For F may be taken to be the support of the measure μ of Theorem 2.1, or all of $E(\phi)$.

3. ALGEBRAS ON PLANE SETS

Let $\mathbb{C}$ be the complex plane and K a compact subset of $\mathbb{C}$. Let bK and $\operatorname{int} K$ denote, respectively, the boundary and interior of K. Let $\hat{K}$ denote the union of K and the bounded components of $\mathbb{C} - K$.

We shall be interested in the following three algebras in this section.

The algebra $P(K)$ consists of the functions in $C(K)$ which can be approximated uniformly on K by polynomials in z.

The algebra $R(K)$ consists of the functions in $C(K)$ which can be approximated uniformly on K by rational functions with poles off K.

The algebra $A(K)$ consists of the functions in $C(K)$ which are analytic on $\operatorname{int} K$.

Recall that the maximal ideal space of $P(K)$ is $\hat{K}$. The maximal ideal space of either $R(K)$ or $A(K)$ is K. In addition, $\partial_{P(K)} = b\hat{K}$, and $\partial_{R(K)} = \partial_{A(K)} = bK$.

Mergelyan's theorem (see [3, Theorem 9.1, p. 48]) implies that $P(K) = A(K)$ if and only if $K = \hat{K}$, and Runge's theorem implies that $P(K) = R(K)$ if and only if $K = \hat{K}$. A consequence of a theorem of Glicksberg [5, p. 117] is that $R(K) = A(K)$ if and only if $\overline{\operatorname{Re}(R(K))} = \overline{\operatorname{Re}(A(K))}$.

3.1. THEOREM. *Let A stand for any one of $P(K)$, $R(K)$, $A(K)$, or $C(K)$, and let B stand for one of the others. Then $A = B$ if and only if $ba(A) = ba(B)$.*

3.2. LEMMA. *Let A be a uniform algebra on a compact Hausdorff space X.*

Suppose there exist $\beta \in M_A$, a representing measure $m \in M_\beta$, and a function $f \in A$ such that

$$\beta(f) = 0 \qquad \textit{but} \quad 0 \notin f(\mathrm{supp}(m)).$$

Then any function $\phi \in C(X)$ of norm one that agrees with $\bar{f}/|f|$ on supp(m) *belongs to $ba(A)$. In particular $ba(A) \neq \varnothing$.*

Proof. For such a function f, the measure fm belongs to $A^\perp$ and satisfies the conditions of Theorem 2.1. Q.E.D.

3.3. Lemma. *Suppose A is a uniform algebra on X such that $M_A = X$. Let $\phi \in C(X)$ have unit modulus everywhere. By the Arens-Royden Theorem we can write $\phi \in e^{iu}f/|f|$ for some $u \in C_R(X)$ and $f \in A^{-1}$. Then $\phi \in ba(A)$ if and only if*

$$d(u, \mathrm{Re}(A)) \geqslant \pi/2. \tag{3.1}$$

In particular, if A is not Dirichlet on X, then $ba(A) \neq \varnothing$.

Proof. We remark that $\phi \in ba(A)$ if and only if $e^{iu} \in ba(A)$. This follows from Theorem 2.1 and the observation that $\mu \in A^\perp$ if and only if $f\mu \in A^\perp$.

Suppose e^{iu} is not badly approximable. By Theorem 2.2 there exists a function $g \in A$ such that $\mathrm{Re}\, ge^{-iu} > 0$. In particular, g has a continuous logarithm. This implies $g = e^h$ for some $h \in A$ (see, for example, [3, p. 88, Corollary 6.2]). Because $\mathrm{Re}\exp(h - iu) > 0$, there exists an integer-valued, continuous function b such that

$$\| \mathrm{Im}\, h - u - 2\pi b \| < \pi/2.$$

By the Shilov Idempotent Theorem [3, p. 88, Corollary 6.5], $b \in A$. Thus $\mathrm{Im}\, h - 2\pi b \in \mathrm{Re}(A)$ and so (3.1) fails to hold.

On the other hand, if (3.1) fails, choose $g \in A$ with $\| \mathrm{Im}\, g - u \| < \pi/2$. Then $\mathrm{Re}\, e^g e^{-iu} > 0$ so $e^{iu} \notin ba(A)$ by Theorem 2.2. Q.E.D.

Proof of Theorem 3.1. *Case* (i). $A = P(K)$, $B = C(K)$. By Mergelyan's theorem, if $P(K) \neq C(K)$ then int $\hat{K}$ is not empty. Let $\beta \in \mathrm{int}\, \hat{K}$ and let m be a representing measure for β supported on $\partial_A = b\hat{K}$. Let $f(z) = z - \beta$. Then β, m, and f satisfy the conditions of Lemma 3.2, so $ba(P(K)) \neq \varnothing = ba(C(K))$.

Case (ii) $A = A(K)$, $B = C(K)$. If $A(K) \neq C(K)$, then int $K \neq \varnothing$. Then $ba(A) \neq \varnothing$ follows from Lemma 3.2 exactly the way case (i) does.

Case (iii). $A = P(K)$, $B = R(K)$. If $P(K) \neq R(K)$, then $K \neq \hat{K}$. Choose

$\beta \in \hat{K} - K$. The argument of case (i) shows that $\phi \in ba(P(K))$ if ϕ is defined by

$$\phi(z) = \frac{\overline{z - \beta}}{|z - \beta|}, \qquad z \in K.$$

We need only show $\phi \notin ba(R(K))$. Let

$$f(z) = (z - \beta)^{-1}, \qquad z \in K.$$

Then $f \in R(K)$ and $\operatorname{Re} f\bar{\phi} > 0$. By Theorem 2.2, $\phi \notin ba(R(K))$.

Case (iv). $A = R(K)$, $B = C(K)$. If $\operatorname{int} K = \varnothing$ and $R(K) \neq C(K)$, then the theorem of Glicksberg mentioned prior to this theorem implies that $\operatorname{Re}(R(K))$ is not dense in $C_R(K)$. If $\operatorname{int} K \neq \varnothing$, this is automatic. Thus, in either case, $R(K)$ is not Dirichlet on K. Lemma 3.3 then shows that $ba(R(K)) \neq \varnothing$.

Case (v). $A = P(K)$, $B = A(K)$. If $P(K) \neq A(K)$, then $K \neq \hat{K}$ and so $P(K) \neq R(K)$. Case (iii) has shown that $ba(R(K)) \neq ba(P(K))$. Because $ba(A(K)) \subset ba(R(K))$, the present case is proved.

Case (vi). $A = R(K)$, $B = A(K)$. If $R(K) \neq A(K)$, there exists $u \in \operatorname{Re}(A(K))$ such that $u \notin \overline{\operatorname{Re}(R(K))}$. Taking a suitable multiple of u, we may assume $d(u, \operatorname{Re}(R(K))) \geqslant \pi/2$. By Lemma 3.3, $e^{iu} \in ba(R(K))$ but $e^{iu} \notin ba(A(K))$.

This takes care of all cases. Q.E.D.

4. Existence of Badly Approximable Functions

Because of Theorems 2.1 and 2.2 it is often enough to study functions with constant modulus. Let us denote by $U(X)$ the set of functions in $C(X)$ with modulus one everywhere. Denote by $U_e(X)$ the subset of $U(X)$ of functions of the form e^{iu}, $u \in C_R(X)$.

If A is Dirichlet on X and $u \in C_R(X)$, then we can choose $f \in A$ with $\| u - \operatorname{Im} f \| < \pi/2$. This implies

$$\operatorname{Re} e^{f} e^{-iu} > 0,$$

so that $e^{iu} \notin ba(A)$ by Theorem 2.2. The same argument yields the following.

4.1. Proposition. *For any uniform algebra A, $e^{iu} \in ba(A)$ implies $d(u, \operatorname{Re}(A)) \geqslant \pi/2$.*

Corollary. *If $u \in \overline{\operatorname{Re}(A)}$, then $e^{iu} \notin ba(A)$.*

In particular, if A is Dirichlet, then $ba(A) \cap U_e(X) = \varnothing$.

DEFINITION. *For any uniform algebra A, define* $\mathrm{Arg}(A)$ *to be the set*

$$\{u \in C_R(X) \colon \exists f \in A \cap C(X)^{-1} \text{ such that } e^{iu} = f/|f|\}.$$

Note that $\mathrm{Re}(A) \subset \mathrm{Arg}(A)$. For if $g \in A$ and $u = \mathrm{Re}\, g$, then $f = e^{ig}$ belongs to A and $e^{iu} = f/|f|$. We can improve Proposition 4.1 by using $\mathrm{Arg}(A)$ in place of $\mathrm{Re}(A)$.

4.2. LEMMA. *Let $u \in C_R(X)$. Then $e^{iu} \in ba(A)$ if and only if $d(u, \mathrm{Arg}(A)) \geqslant \pi/2$.*

Proof. If $d(u, \mathrm{Arg}(A)) < \pi/2$, choose $f \in A \cap C(X)^{-1}$ to satisfy $f/|f| = e^{iw}$ and $\| u - w \| < \pi/2$. But then $\mathrm{Re}\, fe^{-iu} > 0$ and $e^{iu} \notin ba(A)$.

Conversely, suppose $e^{iu} \notin ba(A)$. Then $\mathrm{Re}\, ge^{-iu} > 0$ for some $g \in A$. This implies $ge^{-iu} = e^h$ for some $h \in C(X)$ satisfying $\| \mathrm{Im}\, h \| < \pi/2$. Clearly $u + \mathrm{Im}\, h \in \mathrm{Arg}(A)$ so

$$d(u, \mathrm{Arg}(A)) \leqslant \| \mathrm{Im}\, h \| < \pi/2. \qquad \text{Q.E.D.}$$

4.3. LEMMA. *If $w \in \mathrm{Arg}(A)$ and $d(w, \mathrm{Re}(A)) < \pi$, then $w \in \mathrm{Re}(A)$.*

Proof. We may, without loss of generality, assume that $\| w \| < \pi$. This is because $\mathrm{Arg}(A) + \mathrm{Re}(A) \subset \mathrm{Arg}(A)$. Choose $f \in A \cap C(X)^{-1}$ such that $f/|f| = e^{iw}$. This implies $f(X)$ is disjoint from the nonpositive real axis and so $\log z$ can be approximated by polynomials uniformly on $f(X)$ (taking, for example, the principal branch of the logarithm). Thus, if $g = \log f$, then $g \in A$. Since $\exp i(\mathrm{Im}\, g - w) = 1$,

$$\mathrm{Im}\, g - w = 2\pi b,$$

where b is a continuous, integer-valued function on X. We will show $b \in A$.

For any integer n, let

$$F_n = \{x \in X \colon b(x) = n\}.$$

Because $\| w \| < \pi$, $F_n = \{x \in X \colon | \mathrm{Im}\, g - 2\pi n | < \pi\}$. It will be enough to show that χ_n, the characteristic function of F_n, is in A. Since g takes each F_n into the strip $| \mathrm{Im}\, z - 2\pi n | < \pi$, there is a sequence of polynomials $\{q_k\}$ which converges uniformly to 1 on $g(F_n)$ and uniformly to 0 on $g(F_m)$ for $m \neq n$. But then $q_k(g)$ converges uniformly to χ_n. Therefore $\chi_n \in A$.

Since $b \in A$, $w = \mathrm{Im}\, g - 2\pi b$ belongs to $\mathrm{Re}(A)$. Q.E.D.

4.4. LEMMA. *Let $Q \subset C_R(X)$ be invariant under multiplication by positive scalars. Let $S = \{u \in Q: e^{iu} \in ba(A)\}$. Then*

$$Q \subset \overline{\mathrm{Re}(A)} \cup \bigcup_{t>0} tS.$$

Proof. Let $u \in Q$ and suppose $u \notin \overline{\mathrm{Re}(A)}$. Let t_0 be a positive scalar, chosen so that

$$d(t_0 u, \mathrm{Re}(A)) = \pi/2. \tag{4.1}$$

Then $e^{it_0 u} \in ba(A)$. For otherwise, by Lemma 4.2, there exists $w \in \mathrm{Arg}(A)$ such that

$$\| t_0 u - w \| < \pi/2. \tag{4.2}$$

This and (4.1) imply $d(w, \mathrm{Re}(A)) < \pi$. By Lemma 4.3, $w \in \mathrm{Re}(A)$. Then (4.2) implies $d(t_0 u, \mathrm{Re}(A)) < \pi/2$, contradicting (4.1). Consequently, $t_0 u \in S$. Q.E.D.

The main idea of the proof is the following. (This should be compared with Proposition 4.1.)

4.5. PROPOSITION. *If $u \in C_R(X)$ and if $d(u, \mathrm{Re}(A)) = \pi/2$, then $e^{iu} \in ba(A)$.*

4.6. THEOREM. *Let A and B be uniform algebras on X. Suppose that*

$$U_e(X) \cap ba(A) \subset U_e(X) \cap ba(B).$$

Then $\mathrm{Re}(B) \subset \overline{\mathrm{Re}(A)}$.

Proof. Replace Q by $\mathrm{Re}(B)$ in Lemma 4.4. We need only show that the corresponding S is empty. But if there exists $u \in \mathrm{Re}(B)$ such that $e^{iu} \in ba(A)$, the hypothesis implies $e^{iu} \in ba(B)$. However,

$$d(u, \mathrm{Arg}(B)) \leqslant d(u, \mathrm{Re}(B)) = 0,$$

contradicting Lemma 4.2. Q.E.D.

The following is obtained by setting $B = C(X)$ in Theorem 4.6.

4.7. COROLLARY. *A uniform algebra A is Dirichlet on X if and only if $U_e(X) \cap ba(A) = \varnothing$.*

A remark about terminology: We will say that two functions $\phi, \psi \in C(X)$ are "homotopic in $\mathbb{C} - \{0\}$" if there is a continuous function $F: X \times [0, 1] \to$

$\mathbb{C} - \{0\}$ such that, for all $x \in X$, $F(x, 0) = \phi(x)$, $F(x, 1) = \psi(x)$. In particular, neither ϕ nor ψ vanish anywhere on X.

4.8. LEMMA. *Let A and B be uniform algebras on X. Suppose $ba(A) \subset ba(B)$. Then for every $g \in B$ and compact set $K \subset X$ satisfying $0 \notin g(K)$, there exists $f \in A$ such that $f|_K$ and $g|_K$ are homotopic in $\mathbb{C} - \{0\}$.*

Proof. Let g and K be as stated. Let $Y = \{x \in X: |g(X)| \geq \inf_K |g|\}$, so that Y is a closed G_δ-set containing K. Choose $h \in C(X)$ so that

$$h(x) = g(x)/|g(x)|, \qquad \text{all } x \in Y,$$
$$|h(x)| < 1, \qquad \text{all } x \notin Y.$$

Then $E(h) = Y$. By Theorem 2.2, $h \notin ba(B)$. By hypothesis, $h \notin ba(A)$, and thus there exists $f \in A$ with $\|f - h\| < 1$. The formulas,

$$F(x, t) = th(x) + (1 - t) f(x), \qquad x \in K, \quad t \in [0, 1],$$
$$G(x, t) = th(x) + (1 - t) g(x), \qquad x \in K, \quad t \in [0, 1],$$

define homotopies from $f|_K$ to $h|_K$ and from $g|_K$ to $h|_K$. It is easily verified that neither F nor G vanish on $K \times [0, 1]$, so $f|_K$ and $g|_K$ are homotopic in $\mathbb{C} - \{0\}$. Q.E.D.

4.9. THEOREM. *For a uniform algebra A on X, $ba(A) = \varnothing$ if and only if both of the following hold*:

(a) *A is Dirichlet on X.*

(b) *For every $\phi \in C(X)$ and compact $K \subset X$ satisfying $0 \notin \phi(K)$, there exists $f \in A$ such that $f|_K$ and $\phi|_K$ are homotopic in $\mathbb{C} - \{0\}$.*

Proof. Suppose $ba(A) = \varnothing$. Then (a) follows from Corollary 4.7 and (b) follows from Lemma 4.8 upon setting $B = C(X)$.

On the other hand, suppose (a) and (b) hold and let $\phi \in C(X)$. Let $K = E(\phi)$ and choose $f \in A$ to satisfy (b). Then $f\bar{\phi}|_K$ is homotopic in $\mathbb{C} - \{0\}$ to a constant function. This implies that $f\bar{\phi}|_K = e^h$ for some $h \in C(K)$. Because of (a), we can choose $g \in A$ satisfying

$$\| \operatorname{Im} g|_K - \operatorname{Im} h \| < \pi/2.$$

Therefore

$$\operatorname{Re} e^{-g(x)} f(x) \overline{\phi(x)} > 0, \qquad \text{all } x \in K.$$

Now $e^{-g} f \in A$, so Theorem 2.2 implies that $\phi \notin ba(A)$. Because ϕ was arbitrary, $ba(A) = \varnothing$. Q.E.D.

Remarks. It follows from Theorem 4.6 that $\overline{\mathrm{Re}(A)} = \overline{\mathrm{Re}(B)}$ whenever $ba(A) = ba(B)$. Thus case (vi) of Theorem 3.1 may be deduced from this observation (which involves no assumptions about M_A or M_B) without using Lemma 3.3.

Theorem 4.9 is not vacuous, for Browder and Wermer [2] have constructed a uniform algebra on an arc X which is Dirichlet on X. The properties of an arc are such that (b) of Theorem 4.9 can be satisfied with $f = 1$, a constant function.

We can obtain from Lemmas 4.2 and 4.3 the following interesting result.

If A is a uniform algebra on X and if for every $e^{iu} \in U_e(X)$ there exists $f \in A \cap C(X)^{-1}$ satisfying $f/|f| = e^{iu}$, then $A = C(X)$.

Indeed, the hypotheses say $\mathrm{Arg}(A) = C_R(X)$. This implies $ba(A) = \varnothing$ so that A is Dirichlet on X. This, with Lemma 4.3, implies in turn that $\mathrm{Re}(A) = \mathrm{Arg}(A) = C_R(X)$. An appeal to Corollary 1 of Hoffman and Wermer [8] yields $A = C(X)$.

This appears to be a new characterization of $C(X)$ and provides a complement to a theorem due to Gorin [7]—at least for metrizable X.

5. Rational Functions on Finite Connected Sets in the Plane

Throughout this section let Y denote a compact subset of $\mathbb{C}$ such that $\mathbb{C} - Y$ has finitely many components. Let bY denote its boundary and int Y its interior. If U is a component of int Y, then the number of components of bU does not exceed the number of components of $\mathbb{C} - Y$.

Let A denote the set of restrictions to bY of the functions in $R(Y)$. Note that $M_A = Y$. A combination of Theorem 3.13 and Lemma 3.6 of Glicksberg [6] yields the following.

5.1. Lemma. *Let $\{U_i\}$ be the set of components of* int Y. *For each i, let λ_i be harmonic measure on bU_i for some point in U_i. Let $\mu \in A^\perp$. Then there is a unique decomposition*

$$\mu = \sum \mu_i ,$$

where $\mu_i \in R(\overline{U}_i)^\perp$, $\mu_i \ll \lambda_i$, and $\sum \|\mu_i\| = \|\mu\|$.

5.2. Lemma. *Let $\phi \in C(bY)$. Then $\phi \in ba(A)$ if and only if there exists a component U of* int Y *such that $|\phi|_{bU}| = \|\phi\|$ and such that $\phi|_{bU} \in ba(R(\overline{U}|_{bU}))$.*

Proof. Suppose ϕ satisfies these conditions. Then for every $r \in R(\overline{U})$,

$\| \phi - r \|_{bU} \geqslant \| \phi \|$. Therefore, since the restrictions to bU of functions in A belong to $R(\overline{U})\,|_{bU}$, $\| \phi - f \| \geqslant \| \phi \|$ for every $f \in A$.

On the other hand suppose $\phi \in ba(A)$. Choose nonzero $\mu \in A^{\perp}$ such that μ satisfies $\operatorname{supp}(\mu) \subset E(\phi)$ and $\phi\mu \geqslant 0$. Write $\mu = \sum \mu_i$ as above. Then we also have $\operatorname{supp}(\mu_i) \subset E(\phi)$ and $\phi\mu_i \geqslant 0$. Consequently, we may assume that $\mu \in (R(\overline{U})\,|_{bU})^{\perp}$ for some component U of int Y and that $\mu \ll \lambda$, where λ is harmonic measure on bU for some point $\beta \in U$.

It remains to be shown that $| \phi |_{bU} |$ is constant. For this, it suffices to show that $\operatorname{supp}(\mu) = bU$. If this is not the case, then we can find a disk D with $D \cap bU \neq \varnothing$ and $D \cap \operatorname{supp}(\mu) = \varnothing$. Consider the function,

$$\hat{\mu}(s) = \int (z - s)^{-1}\, d\mu\,(z)$$

defined and analytic off $\operatorname{supp}(\mu)$. Because $\mu \in R(\overline{U})^{\perp}$, $\hat{\mu}$ vanishes off $\overline{U}$. But the disk D must meet both $\mathbb{C} - \overline{U}$ and U. Since $\hat{\mu}$ is analytic on $U \cup D \cup (\mathbb{C} - \overline{U})$, it must vanish also on U. This implies (see, for example, [3, Theorem 8.1, p. 46]) $\mu \in R(bY)^{\perp}$. But $R(bY) = C(bY)$. (For, by Theorem 3.13 of [6], the points of bY are trivial Gleason parts for $R(Y)$ and so also for $R(bY)$. Thus bY is the minimal boundary for $R(bY)$ and we can apply Bishop's criterion [1, Theorem 4] to obtain $R(bY) = C(bY)$.) This contradicts $\mu \neq 0$.

Q.E.D.

Let $\phi \in ba(A)$ and let U be a component of int Y satisfying the conditions of this lemma. Let $X = bU$ and $B = R(\overline{U})\,|_{bU}$ so that B is a uniform algebra on X. We have then $\phi\,|_X \in ba(B)$ and $| \phi\,|_X | = \| \phi \|$. Thus we need only investigate unimodular functions in $ba(B)$.

For any $g \in C(X)^{-1}$ there exists a unique integer m with the following property: For any $\alpha \in U$ there exists a function $r \in B^{-1}$ such that g is homotopic in $\mathbb{C} - \{0\}$ to $(z - \alpha)^m\, r$. This may be shown by applying the Arens–Royden theorem to $R(X)$. We call m the index of g and write $m = \operatorname{ind}(g)$. It is clear that $\operatorname{ind}(g) = \operatorname{ind}(h)$ if g and h are homotopic in $\mathbb{C} - \{0\}$. Moreover, elements of $B \cap C(X)^{-1}$ have nonnegative index.

This coincides with the definition used in the Introduction when $X = bU$ consists of a finite union of disjoint simple Jordan curves. To see this, suppose $g, h \in C(X)^{-1}$, $r \in R(\overline{U})^{-1}$, and $\alpha \in U$ satisfy

$$g(z) = (z - \alpha)^m\, r(z)\, e^{h(z)}, \qquad z \in X.$$

The change in argument of $r(z)$ as z travels along X once in the positive direction is zero because r has no poles or zeros in U. The change in argument of $e^{h(z)}$ is zero. Thus, the change in argument of $g(z)$ is equal to that of $(z - \alpha)^m$, or $2\pi m$.

Let τ be a positive measure on X. Define $H^2 = H^2(\tau)$ to be the closure of B in $L^2(\tau)$. For $\phi \in L^\infty(\tau)$ define the *Toeplitz operator* T_ϕ on H^2 by

$$T_\phi f = P(\phi f), \qquad f \in H^2,$$

where P is the orthogonal projection of L^2 onto H^2. Now $\phi \to T_\phi$ is a contractive linear mapping from L^∞ to $L(H^2)$, the bounded operators on H^2. It is evident that $BH^2 \subset H^2$ and the following formulas hold:

$$(T_\phi)^* = T_{\bar\phi}\,; \tag{5.1}$$

$$\text{for all } f \in B, \qquad T_{\phi f} = T_\phi T_f\,; \tag{5.2}$$

$$\text{for all } f \in B, \qquad T_{\phi \bar f} = T_{\bar f} T_\phi\,. \tag{5.3}$$

The three lemmas that follow and their proofs are the analogs for the present context of Lemmas 7.1, 7.2, and 7.3 in [4]. Some changes have to be made because of the omission of the assumption of smoothness for bU. However, the main change is that Lemma 5.5 here is much weaker than Lemma 7.3 in [4]. It is not clear how much of Lemma 7.3 generalizes to the present case.

5.3. LEMMA. *Let $\alpha \in U$ and let τ be a positive measure on X. Suppose $(z-\alpha)\,H^2(\tau)$ has codimension 1 in $H^2(\tau)$. Then $T_\phi T_\psi - T_{\phi\psi}$ is a compact operator whenever $\phi, \psi \in C(X)$. Furthermore, if $\phi \in C(X)$ does not vanish on X, then T_ϕ is a Fredholm operetor and*

$$\operatorname{ind}(\phi) = -\operatorname{index} T_\phi\,.$$

Proof. Here

$$\operatorname{index} T_\phi = \dim \mathscr{N}(T_\phi) - \operatorname{codim} \mathscr{R}(T_\phi),$$

where $\mathscr{N}$, $\mathscr{R}$ denote "nullspace" and "range," respectively. Now $T_\phi T_\psi - T_{\phi\psi} = 0$ when $\psi \in B$ by (5.2). If $\psi(z) = 1/(z-\alpha)^n$ for some positive integer n, then $T_\phi T_\psi - T_{\phi\psi} = 0$ on $(z-\alpha)^n\,H^2$ and so is at most n-dimensional. By Runge's theorem, the linear combinations of functions in B and the functions $1/(z-\alpha)^n$, $n > 0$, are dense in $R(X)$. But, as in the proof of Lemma 5.2, $R(X) = C(X)$. Thus $T_\phi T_\psi - T_{\phi\psi}$ is compact for all $\phi, \psi \in C(X)$.

If ϕ does not vanish on X, take $\psi = \phi^{-1}$ in the above to see that T_ϕ is Fredholm.

Let $m = \operatorname{ind}(\phi)$. For $\alpha \in U$ fixed, write

$$\phi(z) = (z-\alpha)^m\, r(z)\, h(z),$$

where $r \in B^{-1}$ and h has continuous logarithm on X. Now T_r is invertible, so index $T_r = 0$. Any homotopy in $\mathbb{C} - \{0\}$ from h to 1 yields a path from T_h to I in the Fredholm operators, so index $T_h = 0$. Therefore,

$$\text{index } T_\phi = \text{index } T_{(z-\alpha)^m}.$$

If $m \geqslant 0$, index T_ϕ is easily seen to be $-m$. If $m < 0$, then $-m = \text{ind}(\bar{\phi}) > 0$ and we obtain index $T_\phi = -\text{index } T_{\bar{\phi}} = -m$. Q.E.D.

5.4. LEMMA. *Let $\phi \in C(X)$ be unimodular. Then ϕ is badly approximable if and only if there is a positive measure τ on X such that $(z - \alpha) H^2(\tau)$ has codimension 1 in $H^2(\tau)$ for any $\alpha \in U$ and such that $T_\phi 1 = 0$.*

Proof. If $\phi \notin ba(B)$, there is $f \in B$ with $\| f - \phi \| < 1$. Then, because ϕ is unimodular, $\| 1 - \bar{f}\phi \| < 1$. Therefore

$$\| I - T_{\bar{f}} T_\phi \| = \| T_{1-\bar{f}\phi} \| < 1$$

by (5.3), and so $T_{\bar{f}} T_\phi$ is invertible. In particular $T_\phi 1 \neq 0$.

If $\phi \in ba(B)$, choose a nonzero $\mu \in B^\perp$ with $\phi\mu \geqslant 0$. Let $\tau = \phi\mu$. Now $\int f\bar{\phi}\, d\tau = \int f\, d\mu = 0$, for all $f \in B$, and so $\phi \perp H^2(\tau)$. But this is equivalent to $T_\phi 1 = P(\phi) = 0$.

To prove that $(z - \alpha) H^2(\tau)$ has codimension 1 in $H^2(\tau)$, suppose $(z - \alpha) H^2 = H^2$. Then $1/(z - \alpha)^n \in H^2$ for any integer $n \geqslant 0$. Thus $\int 1/(z - \alpha)^n \bar{\phi}\, d\tau = 0$, for all $n \geqslant 0$, and so μ annihilates the linear span of the functions in B and $1/(z - \alpha)^n$, $n \geqslant 0$. By Runge's theorem, $\mu \in R(X)^\perp = C(X)^\perp = \{0\}$. This contradicts the choice of μ. Thus $(z - \alpha) H^2$ has codimension at least 1. Since $(z - \alpha) B$ has codimension 1 in B, $(z - \alpha) H^2$ has codimension exactly 1. Q.E.D.

5.5. LEMMA. *Let τ be a positive measure on X. If $\phi \in C(X)$ is unimodular and satisfies $T_\phi 1 = 0$, then $\dim \mathcal{N}(T_{\bar{\phi}}) \leqslant N$, where $N + 1$ is the number of components of bU.*

Proof. Let $f \in \mathcal{N}(T_{\bar{\phi}}) \subset H^2(\tau)$. Now $T_{\bar{\phi}} f = 0$ implies

$$\int f\bar{\phi}h\, d\tau = 0, \qquad \text{all } h \in B.$$

And from $T_\phi 1 = 0$ we can conclude that

$$\int \bar{\phi} f h\, d\tau = 0, \qquad \text{all } h \in B.$$

Thus $f\bar{\phi}\, d\tau$ annihilates both B and its conjugate: $f\bar{\phi}\, d\tau \in \text{Re}(B)^\perp$. But $\text{Re}(B)^\perp$

is the span of the real measures in $B^\perp$ and, by Lemma 5.1, $B^\perp$ is just the set of measures in $A^\perp$ that are absolutely continuous with respect to harmonic measure on X for points in U. By Glicksberg's Theorem 3.13 in [6], $\mathrm{Re}(B)^\perp$ has dimension at most N. Thus $\dim \mathcal{N}(T_{\bar\phi}) \leqslant N$. Q.E.D.

5.6. THEOREM. *Let $N+1$ be the number of components of bU. If $\phi \in ba(B)$, then ϕ is unimodular and* $\mathrm{ind}(\phi) < N$. *On the other hand, if $\phi \in C(X)$ is unimodular and* $\mathrm{ind}(\phi) < 0$, *then $\phi \in ba(B)$.*

Proof. Let $\phi \in ba(B)$. By Lemma 5.2, ϕ is unimodular. By Lemma 5.4 there exists a positive measure τ on X such that $(z-\alpha)\,H^2(\tau)$ has codimension 1 in $H^2(\tau)$ for $\alpha \in U$, and such that $T_\phi 1 = 0$. By Lemma 5.3

$$\mathrm{ind}(\phi) = -\mathrm{index}\ T_\phi = \dim \mathcal{N}(T_{\bar\phi}) - \dim \mathcal{N}(T_\phi).$$

By Lemma 5.5, $\dim \mathcal{N}(T_{\bar\phi}) \leqslant N$ and so, because $1 \in \mathcal{N}(T_\phi)$, $\mathrm{ind}(\phi) < N$.

Suppose $\phi \in U(X)$ and $\mathrm{ind}(\phi) < 0$. If there exists $f \in B$ with $\| \phi - f \| < 1$, then ϕ and f are homotopic in $\mathbb{C} - \{0\}$. Thus $\mathrm{ind}(f) < 0$, a contradiction. Q.E.D.

If $\mathbb{C} - Y$ is connected, then U must be simply connected and the integer N in the preceding theorem is 0. Combining this with Lemma 5.2 yields the following extension of Poreda's theorem.

5.7. COROLLARY. *Let $Y \subset \mathbb{C}$ be compact with $\mathbb{C} - Y$ connected and let $A = P(bY)$ $(= R(Y)|_{bY})$. Let $\phi \in C(bY)$ have norm 1. Then $\phi \in ba(A)$ if and only if there exists a component U of* int *Y such that $\phi|_{bU}$ is unimodular and* $\mathrm{ind}(\phi|_{bU}) < 0$.

6. GENERALIZED ANALYTIC FUNCTIONS

Let Γ be a discrete, totally ordered, nontrivial Abelian group satisfying the following:

$$\{\gamma \in \Gamma : \gamma \geqslant 0\} \equiv \Gamma^+ \text{ is a semigroup} \quad \text{and} \quad -\Gamma^+ \cap \Gamma^+ = \{0\}; \tag{6.1}$$

$$\gamma_1 \geqslant \gamma_2 \text{ if and only if } \gamma_1 - \gamma_2 \in \Gamma^+. \tag{6.2}$$

Let G be the group of characters of Γ with the topology of pointwise convergence on Γ. Then G is compact and we can identify Γ with a (multiplicative) subgroup of $C(G)$ by letting γ correspond to f_γ defined by

$$f_\gamma : x \in G \to x(\gamma).$$

Define $A(G)$ to be the closed algebra generated by $\{f_\gamma : \gamma \in \Gamma^-\}$. It is known (see, for example, [3, Chap. VII]) that $A(G)$ is Dirichlet on G, that normalized Haar measure is multiplicative on $A(G)$, and that $C(G)$ is the only uniform algebra on G properly containing $A(G)$ (that is, $A(G)$ is *maximal*).

6.1. THEOREM. *Let $\phi \in C(G)$. Then $\phi \in ba(A(G))$ if and only if $|\phi|$ is constant and ϕ is homotopic in $\mathbb{C} - \{0\}$ to f_γ for some $\gamma < 0$.*

Poreda's theorem is obtained by taking the integers for Γ.

The proof requires the following lemmas.

6.2. LEMMA. *For every unimodular $\phi \in C(G)$ there exists $\gamma \in \Gamma$ such that ϕ and f_γ are homotopic in $\mathbb{C} - \{0\}$.*

It would be very surprising if this were not a known result or did not follow easily from some theorem of algebraic topology. However, we have been unable to verify this. The proof begins by viewing Γ as a direct sum of copies of the integers (Γ is torsion free) and so G as a product of circles. An index of ϕ in each direction can be defined, and all but finitely many of these are 0. This determines a $\gamma \in \Gamma$ and it can then be shown that $f_\gamma \bar{\phi} = e^{iu}$ for some $u \in C_R(G)$. The details may be found in the thesis [9].

6.3. LEMMA. *Let A be an algebra that is Dirichlet on the compact Hausdorff space X. Let $\phi \in C(X)$ be unimodular. Then $\phi \in ba(A)$ if and only if ϕ is not homotopic in $\mathbb{C} - \{0\}$ to any function in A. As a consequence, if $\phi \in ba(A)$ and $\psi \in U(X)$ is homotopic to ϕ in $\mathbb{C} - \{0\}$, then $\psi \in ba(A)$.*

Proof. Suppose ϕ is not badly approximable. Then there exists $f \in A$ such that $\| \phi - f \| < 1$. Clearly,

$$F(x, t) = tf(x) + (1 - t)\,\phi(x), \qquad x \in X, \quad t \in [0, 1],$$

defines a homotopy in $\mathbb{C} - \{0\}$ from ϕ to f.

Conversely, suppose ϕ is homotopic to $f \in A$. Then $\bar{f}\phi = e^h$ for some $h \in C(X)$. By Corollary 4.7, $\exp(i \operatorname{Im} h)$ is not badly approximable, so there exists $g \in A$ satisfying $\operatorname{Re} gf\bar{\phi} > 0$. Since $gf \in A$, ϕ is not badly approximable.

Q.E.D.

Proof of Theorem 6.1. Suppose $\phi \in U(G)$ and ϕ is homotopic in $\mathbb{C} - \{0\}$ to f_γ, $\gamma < 0$. By Lemma 6.3, we need only show $f_\gamma \in ba(A(G))$. Let m be normalized Haar measure on G. Then, because $f_{-\gamma} \in A(G)$ and m is multiplicative on $A(G)$, we have

$$\int gf_{-\gamma}\, dm = \int g\, dm \int f_{-\gamma}\, dm = 0.$$

Therefore $f_{-\gamma}\, dm \in A(G)^{\perp}$. Since $f_\gamma f_{-\gamma}\, dm \geqslant 0, f_\gamma \in ba(A(G))$ by Theorem 2.1.

Conversely, suppose ϕ is badly approximable, and suppose it has already been shown that ϕ is unimodular. Then, by Lemma 6.2, ϕ is homotopic in $\mathbb{C} - \{0\}$ to f_γ for some $\gamma \in \Gamma$. By Lemma 6.3, $f_\gamma \notin A$, and so $\gamma \notin \Gamma^+$. That is, $\gamma < 0$.

It remains to be shown that $\phi \in U(G)$. Choose a nonzero measure $\mu \in A(G)^{\perp}$ such that $\phi\mu \geqslant 0$ and $\text{supp}(\mu) \subset E(\phi)$. We need only show $\text{supp}(\mu) = G$. Suppose $K \equiv \text{supp}(\mu) \neq G$. Define

$$B = \{ g \in C(G) \colon g\,|_K \in A(G)_K \}.$$

Then B consists of all continuous functions on G whose restrictions to K can be approximated uniformly by functions in $A(G)$. By maximality, $A(G) = B$. In particular, every function which vanishes on K is in $A(G)$. This implies that every measure in $A(G)^{\perp}$ is supported on K. But this is not true of $f_\gamma\, dm$ for $\gamma > 0$. Since we suppose Γ is nontrivial, this is a contradiction and we conclude $\text{supp}(\mu) = G$. Q.E.D.

7. Remarks

(1) Lemma 6.3 can be extended to non-Dirichlet algebras in the following form:

If $\phi \in C(X)$ is unimodular, then $\phi \in ba(A)$ if and only if

$$d(\arg \phi\bar{f}, \operatorname{Arg}(A)) \geqslant \pi/2 \tag{7.1}$$

for every $f \in A \cap C(X)^{-1}$ and every continuous determination of $\arg \phi\bar{f}$.

This condition is vacuously satisfied if no such determination exists for any $f \in A$. When A is Dirichlet, (7.1) is never satisfied, and so $\phi \in ba(A)$ if and only if $\arg \phi\bar{f}$ is never continuous. But this is equivalent to Lemma 6.3.

(2) An argument similar to the one used in the proof of Theorem 6.1 yields the following:

If A is a maximal, essential uniform algebra on X, then every element of $ba(A)$ is unimodular.

An algebra is *essential* if $\bigcup_{\mu \in A^{\perp}} \text{supp}(\mu)$ is dense in X. Combining this with Lemma 6.3, we see that Poreda's theorem is a consequence of the uniform algebra properties of the disk algebra and the topological nature of the circle, and does not require any of the analytic characteristics of polynomials except insofar as they contribute to these properties.

(3) It would be of interest to obtain a result analogous to Theorem 3.1, even if only for $P(X)$ and $C(X)$, when X is a compact set in complex n-space.

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